GROUP THEORY 2024 - 25, EXERCISE SHEET 8

Exercise 1. (hard) *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. (easy) Are the following groups solvable? If so then write down a subnormal series with abelian quotients for each of them:

(1) $\mathbb{Z}$
(2) S_3
(3) S_4
(4) D_{2n} for all n
(5) $G \times H$, where G and H are solvable groups

Exercise 3. (easy) Let H and G be nilpotent groups. Show that $H \times G$ is also a nilpotent group.

Exercise 4. (easy) Let G be any group. Recall that an automorphism of a group G is just an isomorphism $G \to G$. We say that a subgroup $H \subseteq G$ is a characteristic subgroup of G if $\varphi(H) = H$ for all automorphisms φ of G.

(1) Show that a characteristic subgroup of G is always a normal subgroup.
(2) Show that the center $Z(G)$ is a characteristic subgroup of G.
(3) Show that the commutator subgroup $[G, G]$ is a characteristic subgroup of G.

Exercise 5. (medium) Let G be a finite p-group. Show that G is solvable.

Exercise 6. (medium)

(1) Let G be a simple group which is also solvable. Show that G must be abelian. Conclude that G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.
(2) Show that a solvable group having a composition series is necessarily finite.
(3) Show that a finite group is solvable if and only if its composition factors are cyclic groups of prime order.

Exercise 7. (medium) Let k be any field. Consider the group B of 2×2 invertible upper-triangular matrices over k:

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in k^\times, b \in k \right\}.$$

Show that B is a solvable group by exhibiting a subnormal series for B with abelian quotients.

Exercise 8. (hard) The goal of this exercise is to show that $A_n \triangleleft S_n$ is the unique non trivial normal subgroup of S_n for $n \geq 5$. Let $1 \neq H \triangleleft S_n$ be a non-trivial normal subgroup.

(1) Show that the center of S_n is trivial for $n \geq 3$.
  Hint: Write $\alpha \in S_n$ as a product of disjoint cycles and construct $\beta \in S_n$ such that $\alpha^{-1}\beta\alpha \neq \beta$.
(2) Suppose that $H \cap A_n = 1$. Show that H contains an element σ of order 2, and that it is in fact the only non-trivial element, i.e. H is a subgroup of order 2.
(3) Using the first point, find a contradiction by showing that $\sigma = 1$.
(4) Conclude.